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'Holey sheets' — pfaffians and subdeterminants as D-brane operators in large-N gauge theories

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Abstract: In the AdS/CFT correspondence, wrapped D3-branes (such as "giant gravitons") on the string theory side of the correspondence have been identified with pfaffian, determinant and subdeterminant operators on the field theory side. We substantiate this identification by showing that the presence of pairs of such operators in a correlation function of a large-N gauge theory naturally leads to a modified 't Hooft expansion including also worldsheets with boundaries. This happens independently of supersymmetry or conformal invariance.

Keywords: 1/N Expansion, AdS-CFT and dS-CFT Correspondence.

*Incumbent of the Joseph and Celia Reskin career development chair.
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Contents

1. Introduction and summary of results

The correspondence [1]–[4] between AdS_5 backgrounds of type-IIB string theory and four dimensional gauge theories provides the first explicit realization (above two dimensions) of 't Hooft's general identification [5] of large-N gauge theories with string theories, with the string coupling scaling as $g_s \propto 1/N$. The first examples of this correspondence, and most of the cases that have been studied up to now, involved field theories with fields only in adjoint or bifundamental representations. Such field theories naturally map to closed string theories [5]. Backgrounds including D-branes filling AdS_5, or filling AdS_p subspaces of AdS_5, were also studied. In such backgrounds there is also an open string sector on the string theory side of the correspondence, corresponding to having additional fields in the fundamental representation on the field theory side.

However, D-branes (and, therefore, an open string sector) can appear also in the closed string theories corresponding to field theories without any fundamental representation fields. In this paper we will be interested only in localized D-branes, behaving like particles in anti-de Sitter (AdS) space, which should correspond to local operators on the field theory side of the correspondence. The first example of this was given (and its field theory dual identified with a pfaffian operator) in [6], and many other examples have been discovered since then.

A large class of examples of localized D-branes is related to "giant gravitons". One of the most interesting results of the AdS/CFT correspondence is a concrete realization of the general expectation that high momentum excitations in gravity are described by large macroscopic objects, rather than by short wavelength field excitations. It was shown in [7] that gravitons with large angular momentum on the S^5 become D3-branes wrapped on an S^3 inside the S^5 (similar phenomena occur with M-branes in M-theory AdS backgrounds). This example of Myers' dielectric effect [8] goes under the name of "giant gravitons" [7], and it is believed to be part of a broader UV/IR mixing conspiracy in quantum gravity. These states were conjectured in [9] to correspond to subdeterminant operators in the corresponding $\mathcal{N}=4$ SYM theory. We will call the field theory operators corresponding to D-brane states "D-brane-type operators".

In this paper we attempt to understand how open strings can arise in a theory involving only fields in the adjoint representation, whose large-N expansion is naively given by a purely closed string theory. We start by mapping string theory diagrams corresponding to correlation functions of closed string vertex operators in the presence of the D-brane to appropriately normalized correlation functions on the field theory side (we could also similarly discuss open string vertex operators, but we will not do this here). We argue that the consistency of the AdS/CFT correspondence requires that appropriately normalized correlation functions involving pairs of D-brane-type operators, such as pfaffians and subdeterminants in the $\mathcal{N}=4$ SYM theory, should have an expansion in terms of open and closed worldsheets.[1] The bulk of the paper is devoted to showing that such an expansion indeed exists and, moreover, that it exists for pfaffian and subdeterminant operators in any large-N theory, not necessarily supersymmetric or conformal.

Our method involves developing the large-N expansion for (appropriately normalized) correlation functions involving pairs of pfaffian or subdeterminant operators in SO($2N$) and SU(N) theories, respectively. Since these operators involve a number of fields of order N, this expansion is quite different from the usual 't Hooft expansion involving purely closed worldsheets (though, of course, the "interior" of Feynman diagrams, far away from the D-brane-type operators, still maps to closed string worldsheets in the usual way). In particular, we show that boundaries can arise in the worldsheets of the Feynman diagrams,

[1] An alternative way of seeing open strings in the $\mathcal{N}=4$ SYM theory, by analyzing small fluctuations around D-brane-type operators in the BMN limit [10] of the theory, was discussed in [11].

associated with external lines emanating from the D-brane-type operators. We argue that we can map any connected Feynman diagram to a worldsheet, which can have an arbitrary number of boundaries and can also be disconnected. Our main result is the computation of the power of N associated with these Feynman diagrams in appropriately normalized correlation functions. We show that it is precisely given (at leading order in $1/N$) by N^{χ} where χ is the Euler characteristic of the corresponding worldsheet, as expected for a string theory involving open and closed strings whose string coupling scales as $g_s \simeq 1/N$. Our result depends on an assumption that we need to make regarding the existence of a certain contour appearing in a contour integration that arises in the correlation function computation. We believe that this assumption is correct, but we have not been able to prove it beyond the level of the free field theory. It would be interesting to prove this assumption and to make our results more rigorous. It would also be interesting to analyze the $1/N$ corrections to our results.

Another prediction from string theory is that when we map a certain connected diagram to a disconnected string theory worldsheet, its value should factorize into the product of diagrams mapping separately to each component of the worldsheet. We show that this 'factorization' property indeed holds, providing additional evidence for our identification of the mapping of the Feynman diagrams to string worldsheets.

The outline of the paper is the following. In section 2 we review some known results regarding D-branes and "giant gravitons" in the AdS/CFT correspondence. In section 3 we specify the class of field theory quantities that we expect to map to correlation functions of closed strings in the presence of a D-brane, and thus to have an appropriate $1/N$ expansion. In section 4 we analyze the 't Hooft large-N expansion for appropriate correlation functions involving the pfaffian operators in SO($2N$) theories. In section 5 we carry over the analysis to the determinant and subdeterminant operators of SU(N) theories. We show that the appropriately normalized correlation functions behave as expected for a theory involving both closed and open strings. In all our field theory computations we do not use supersymmetry, except in section 4.5 where we discuss special properties of theories like the $\mathcal{N} = 4$ SYM theory in which 2-point functions are not renormalized.

2. D-branes in the AdS/CFT correspondence

The simplest operators to describe in the 't Hooft large-N limit of gauge theories [5] are gauge-invariant operators made of a finite number of basic fields, which remains constant as $N \to \infty$. An example of such operators is $\mathrm{tr}(X^l)$ if X is a field in the adjoint representation of the gauge group. From the construction of the 't Hooft large-N limit it is clear that such operators (if they are "single-trace" operators, namely they cannot be written as a product of two or more gauge-invariant operators) map to local operators on the worldsheet of the string.

Indeed, this is exactly how things work in the AdS/CFT correspondence [1]–[4], which, for instance, relates the SU(N) $\mathcal{N} = 4$ SYM theory to type-IIB string theory on $AdS_5 \times S^5$, and the SO(N) $\mathcal{N} = 4$ SYM theory to an orientifold of type-IIB string theory on $AdS_5 \times RP^5$ [6]. Gauge-invariant operators in the $\mathcal{N} = 4$ SYM theory involving $l \ll N$ basic fields, which are not a product of smaller gauge-invariant operators, are mapped to integrated local vertex operators on the string worldsheet. In particular, those single-trace chiral primary operators of the $\mathcal{N}=4$ SYM theory which are made of a small number of basic fields (or, equivalently, the ones in small representations of the $SU(4)_R$ global symmetry group) are mapped to type-IIB supergravity fields.

The description of such operators as local operators on the string worldsheet is valid only for $l \ll N$. For instance, the operator $\mathrm{tr}(X^{N+1})$ in an $SU(N)$ gauge theory may be written as a linear combination of products of lower order traces, so it should not correspond to an independent vertex operator in the theory, but it is not known how to see this directly in string theory.[2] The usual 't Hooft large-N expansion seems to break down when used for operators involving a large number of fundamental fields (although techniques exist for some baryonic operators, as in [13]), and apriori it is not clear that their correlation functions should have any reasonable large-N expansion. In particular, trace-type operators $\mathrm{tr}(X^l)$ with large l mix significantly with multi-trace operators.[3]

Some particular operators involving a large number of basic fields have been identified with D-branes in the type-IIB background which are localized in AdS_5 (we will only discuss localized D-branes in this paper). The D-brane states that were matched with the field theory side fall into two classes. One class includes D-branes which are topologically stable on the string theory side. This means that there is some charge in the theory and that the D-brane state is the lightest state with this charge, ensuring its stability. One can then usually identify the operator in the field theory side by looking for the operator of lowest dimension with the same charge (in supersymmetric theories these usually turn out to be chiral operators in short representations of the superconformal algebra, which restricts the operator mixing and simplifies the identification). Another class includes D-brane states which are dynamically stable, such as "giant gravitons". One is typically less sure about the identification of these states in the field theory. One of the goals of this paper is to support existing proposals for the identification of the dynamically stable branes.

Topologically stable branes. The first identification of a D-brane with a field theory operator was presented in [6]. In that case the identification involved the pfaffian operator of $\mathcal{N}=4$ supersymmetric $SO(2N)$ gauge theories, defined by

$$\mathrm{Pf}(X) = \frac{1}{(2N)!}\epsilon^{i_1 i_2 \cdots i_{2N}} X_{i_1 i_2} X_{i_3 i_4} \cdots X_{i_{2N-1} i_{2N}} \,, \tag{2.1}$$

where the X's can be any of the six adjoint scalar fields in the theory, contracted in their $SO(6)_R$ indices in a symmetric traceless manner so as to form a chiral primary operator. This was identified with a D3-brane wrapped on the non-trivial 3-cycle in the orientifold RP^5. This identification was motivated by the fact that this brane carries a $\mathbb{Z}_2$ charge (coming from the fact that the relevant cohomology of RP^5 is $\mathbb{Z}_2$) which may be identified

[2] Presumably this is related to our poor understanding of string theory in RR backgrounds; the analogous bound is understood in cases which only involve NS-NS backgrounds, as in [12].

[3] A basis of operators which have diagonal 2-point functions in free field theory was constructed in [14]. For $l \simeq \sqrt{N}$ this mixing plays an important role in the recent study of string theory in plane wave backgrounds.

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with the $\mathbb{Z}_2$ charge in the center of the SO($2N$) gauge theory, under which the pfaffian operator is charged (but operators including less X's are not). It was supported by various computations, including the mass of this D3-brane (related to the conformal dimension of the corresponding operator), its SU$(4)_R$ transformation properties, etc.

Similar operators, corresponding to D3-branes wrapped on non-trivial 3-cycles, were found to exist in other AdS_5 backgrounds as well. In $AdS_5 \times M$ backgrounds corresponding to product gauge groups with bi-fundamental matter fields, it was found [15]–[19] that D3-branes wrapped on 3-cycles of M could be identified with dibaryon-type operators made of N bi-fundamental fields. We will not discuss this case here, but we expect it to behave similarly to the pfaffian case which we will discuss in detail.

Dynamically stable branes. An example of dynamically stable (supersymmetric) branes in $AdS_5 \times S^5$ was given in [7]. In this case they are D3-branes wrapping a 3-sphere in S^5, which is topologically trivial, and the branes are dynamically stabilized by their angular momentum (which includes contributions both from orbital angular momentum and from interactions with background fields). The size of these branes grows as their angular momentum increases, reaching the maximum possible size for an S^3 in S^5 when the angular momentum reaches the maximum possible value of N (for a single-trace operator). It was argued in [7] that it is in terms of these branes, rather than fundamental strings, that one should describe Kaluza-Klein gravitons of large angular momentum on S^5, and they are therefore called "giant gravitons".

In [9] it was suggested that these D-branes should be identified with determinant and subdeterminant operators of the $\mathcal{N} = 4$ SYM theory, and that the same identification applies also to similar theories with lower supersymmetry (although the details may vary from case to case). For the $\mathcal{N} = 4$ theory, the giant gravitons with angular momentum $L \leq N$ ($N - L \ll N$) were identified with the subdeterminant operators

$$\det{}_L(X) = \frac{1}{L!(N-L)!} \epsilon_{i_1 \cdots i_L i_{L+1} \cdots i_N} \epsilon^{j_1 \cdots j_L i_{L+1} \cdots i_N} X^{i_1}_{j_1} \cdots X^{i_L}_{j_L} \,, \tag{2.2}$$

where X is one of the complex scalar fields of the theory. The subdeterminant operators with $L = 2, 3, \ldots, N$ form an alternative basis to the algebra of gauge-invariant operators, instead of the "standard" $\mathrm{tr}(X^l)$ basis.

Some of the reasons for this identification are [9]:

1. Some correlation functions of these operators are protected and hence can be computed in the free theory. The results can be used to show that these operators form a better basis in the sense that their mixing is smaller by powers of N. The mixing was analyzed more generally in [14], supporting this conclusion.

2. In some cases, several topologically stable branes can be combined to give a dynamically stable brane (for example a pair of pfaffian operators in the SO($2N$) theory), facilitating the identification of the latter.

3. In some cases one can compute the expectation value of these operators along flat directions.

4. This identification naturally explains the bound $L \leq N$ on the angular momentum of the "giant gravitons" coming from D3-branes wrapped on an S^3 in S^5.

In this paper we will support this identification further for the determinant and sub-determinant operators in $\mathrm{SU}(N)$ gauge theory. We expect that a similar analysis will hold for other cases with product gauge groups and adjoint or bi-fundamental matter.

Summary. In summary, D3-branes wrapped on the compact part of space-time in the AdS/CFT correspondence seem to be generally related to gauge-invariant operators formed by multiplying a large number $l \simeq N$ of basic fields in the theory with anti-symmetric contractions. We will analyze the implications of this for the field theory in the next section, focusing on the simplest case of the $\mathcal{N} = 4$ SYM theory with gauge groups $\mathrm{SO}(2N)$ and $\mathrm{SU}(N)$. We will discuss in this paper only D-branes which are completely wrapped on the compact space, so as to give particles on AdS space (which are mapped to local operators in the field theory).[4] There are also various other possible localized D-brane configurations, such as "giant gravitons" extended in AdS_5 [20, 21][5] or D-branes which must have fundamental strings ending on them [6], and we will not discuss these here, though the analysis of the next section should apply also to them.

3. Open string diagrams in field theory

The correlation functions of single-trace operators involving a small number of basic fields in the $\mathcal{N} = 4$ SYM theory map (in the string perturbation theory approximation) to correlation functions of the appropriate closed string vertex operators (for chiral primary operators these can be computed in the supergravity approximation for large $g_{YM}^2 N$). As discussed above, some operators $\mathcal{O}_D$ in the SYM theory are believed to map to D-brane states. So, their correlation functions should correspond to processes involving D-branes. General processes involving D-brane creation and annihilation cannot be studied in string perturbation theory. However, other processes involving D-branes, such as scattering of closed strings off D-branes, can be studied in string perturbation theory, and we will focus on these processes in this paper.

Since we are interested in processes which do not create or annihilate D-branes, we need to have a D-brane present in the initial and final states (for simplicity, we will discuss here the case of a single D-brane). It is simplest to discuss such processes in global AdS space-time, which maps via the AdS/CFT correspondence to the $\mathcal{N} = 4$ SYM theory on $S^3 \times \mathbb{R}$. Of course, correlation functions on $S^3 \times \mathbb{R}$ are related to correlation functions on $\mathbb{R}^4$ by the usual conformal transformation involved in radial quantization. A process involving a D-brane in the initial state maps via this relation to an insertion of the D-brane-type operator $\mathcal{O}_D$ at the origin of $\mathbb{R}^4$, while having the same D-brane in the final state maps to an insertion of $\mathcal{O}_D^\dagger$ at infinity. Of course, we could also have slightly different initial and final D-brane states if we put in different operators at zero and infinity.

[4] D-branes which are not localized but go all the way to the boundary of AdS_5 simply give rise to matter fields in the fundamental representation in the field theory.

[5] A conjecture on the identification of these branes in the field theory was presented in [14].

The simplest processes we can discuss in string perturbation theory are those involving some number of closed string vertex operators in the presence of the D-brane (namely, allowing string worldsheets which have boundaries on the D-brane). The discussion above suggests that if the closed string vertex operators correspond to field theory operators $\mathcal{O}_i(x_i)$, then such a correlation function in closed+open string theory should be related to the field theory correlation function

$$\langle \mathcal{O}_D(0)\mathcal{O}_D^\dagger(\infty)\prod_i \mathcal{O}_i(x_i)\rangle \,. \tag{3.1}$$

What is the exact relation between these correlation functions?[6] In the AdS/CFT correspondence, correlation functions of local operators $\mathcal{O}(x)$ in the CFT are related to the string theory partition function by:

$$\langle e^{\int d^4x\phi_0(x)\mathcal{O}(x)}\rangle_{CFT} = Z_{\text{string}}[\phi(x,z)_{z=0} \simeq \phi_0(x)] \,. \tag{3.2}$$

Here, $\phi_0(x)$ is an arbitrary function, and the field $\phi(x,z)$ is the field in AdS space corresponding to the operator $\mathcal{O}(x)$. Correlation functions $\langle\mathcal{O}(x_1)\ldots\mathcal{O}(x_k)\rangle$ in field theory are the coefficients of $\phi_0(x_1)\ldots\phi_0(x_k)$ in the expansion of the left hand side. The right hand side is the full partition function of string theory with the boundary condition that the field ϕ has the value ϕ_0 (up to an appropriate power of the radial coordinate z) on the boundary of AdS. To get the coefficient of $\phi_0(x_1)\ldots\phi_0(x_k)$ on the right hand side one sums over all closed string topologies with k insertions of the integrated vertex operators of the field ϕ at the appropriate points x_i.

The expansion of $\langle\mathcal{O}(x_1)\ldots\mathcal{O}(x_k)\rangle$ in field theory Feynman diagrams will include disconnected vacuum diagrams. To get rid of them one normalizes by the value of the left hand side for $\phi_0 = 0$, which is just the vacuum partition function. This normalization removes the disconnected closed surfaces with no vertex operator insertions from the string theory expansion of the right hand side. Similarly, to compute (3.1) we will use string theory in a background with a D-brane, and the expansion on the string theory side will include also disconnected surfaces with boundary with no vertex operator insertions, which we would like to remove in the normalization. And, in the field theory side, when computing the generating function $\langle e^{\int d^4x\phi_0(x)\mathcal{O}(x)}\mathcal{O}_D(0)\mathcal{O}_D^\dagger(\infty)\rangle_{CFT}$ it is natural to again normalize by the value of this expression for $\phi_0 = 0$, dividing by $\langle\mathcal{O}_D(0)\mathcal{O}_D^\dagger(\infty)\rangle$.

Thus, we would like to suggest that the relation between correlators of D-brane-type operators in the field theory and computations in a closed+open string theory with boundaries on the corresponding D-brane takes the form

$$\frac{\langle e^{\int d^4x\phi_0(x)\mathcal{O}(x)}\mathcal{O}_D(0)\mathcal{O}_D^\dagger(\infty)\rangle_{CFT}}{\langle\mathcal{O}_D(0)\mathcal{O}_D^\dagger(\infty)\rangle_{CFT}} = Z_{\text{open+closed,D-Brane}}[\phi(x,z)_{z=0} \simeq \phi_0(x)] \,, \tag{3.3}$$

with no vacuum diagrams appearing on the right-hand side. This means that the normalized correlation functions

$$\frac{\langle\mathcal{O}_D(0)\mathcal{O}_D^\dagger(\infty)\prod_i \mathcal{O}_i(x_i)\rangle}{\langle\mathcal{O}_D(0)\mathcal{O}_D^\dagger(\infty)\rangle} \tag{3.4}$$

[6]See [9] and [14] for previous discussions of such correlation functions.

should have a large-N expansion involving surfaces with or without boundaries, all containing vertex operator insertions. Note that in field theory it is easy to compute the denominator of (3.4). The D-brane-type operators diagonalize the 2-point functions and have a fixed dimension Δ_D, so it is simply given by $1/(\infty)^{2\Delta_D}$. This is an infinite factor that will appear also in the numerator of (3.4), so the ratio (3.4) should be finite.

We suggest that finite ratios like (3.4) should map to the correlation function of the closed string vertex operators corresponding to the operators $\mathcal{O}_i$ in the presence of the D-brane. This suggestion agrees with the leading term in the large-N limit of (3.4), which comes from taking a disconnected correlation function $\langle \mathcal{O}_D(0)\mathcal{O}_D^\dagger(\infty)\rangle\langle\prod_i \mathcal{O}_i(x_i)\rangle$ in the numerator, and looking only at planar diagram contributions to $\langle\prod_i \mathcal{O}_i(x_i)\rangle$; this obviously agrees with the leading term in g_s on the right hand side of (3.3), arising from a sphere diagram which does not note the presence of the D-brane. Note that the correspondence works in the simplest way when we normalize the operators $\mathcal{O}_i$ such that their 2-point functions scale as N^2 in the large-N limit, so we will use this normalization throughout this paper (in the normalization we will use, an example of such operators is $N\mathrm{tr}(X^l)$). In this normalization diagrams whose topology has Euler characteristic χ scale as N^χ in the large-N limit.

In string theory, the right-hand side of (3.3) has an expansion in powers of g_s involving even and odd powers of g_s, coming from all diagrams (connected and disconnected, with or without boundaries) appearing in string perturbation theory. Using (3.3), the AdS/CFT correspondence maps this to having a similar good $1/N$ expansion for (3.4) in the $\mathcal{N} = 4$ SYM theory. The identification of the D-brane-type operators is only well-understood in the weakly curved string theory, corresponding to $g_{YM}^2 N \gg 1$. However, we hope that the same properties should hold for any value of $g_{YM}^2 N$, so they should be visible also in perturbation theory (in the 't Hooft limit). In the remainder of the paper we will show (with a mild assumption) that this is in fact true in any large-N gauge theory (not necessarily having a weakly curved string theory dual), though we used supersymmetric theories to motivate it. Note that even though we used conformal invariance of the field theory in the discussion above to motivate the expression (3.4), conformal invariance does not seem to influence the general properties of the large-N expansion (though it certainly affects the space-time dependence of correlation functions), so we could have a similar expansion for (3.4) even in non-conformal theories (with arbitrary positions for the D-brane-type operators), and we will show that this is indeed the case.

4. SO($2N$) gauge theories

In this section we will derive the large-N topological expansion for (3.4) in SO($2N$) gauge theories with pfaffian operators. In these theories, (3.4) takes the form

$$\frac{\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger) N\,\mathrm{tr}(X^{J_1}) N\,\mathrm{tr}(X^{J_2})\cdots N\,\mathrm{tr}({X^\dagger}^{L_1}) N\,\mathrm{tr}({X^\dagger}^{L_2})\cdots\rangle}{\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\rangle}, \tag{4.1}$$

choosing a particularly simple form of $\mathcal{O}_i$ (which should not affect the results) and suppressing the space-time positions of the operators. As discussed above, based on string theory we expect such a large-N expansion to exist, and to include all topologies of compact surfaces with boundaries,[7] at least for the $\mathcal{N}=4$ SYM theory.

The outline of this section is the following. In section 4.1 we will review the known large-N expansion for correlators of single trace operators. In section 4.2 we will analyze the form and N-dependence of a general diagram in the correlator $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\rangle$, and in section 4.3 we will extend the analysis to diagrams in correlators which include trace operators as well. We will identify the topology corresponding to each diagram and find diagrams with all topologies with boundaries. In section 4.4 we will argue that (4.1) indeed has a good large-N expansion, and that the contribution of diagrams which have a specific topology (as defined in section 4.2 and section 4.3) is proportional as expected to N^χ, where χ is that topology's Euler characteristic. The analysis will apply to any SO($2N$) gauge theory, not just supersymmetric ones. In section 4.5 we will discuss some special features of the $\mathcal{N}=4$ SYM theory.

4.1 Review and definitions

We are interested in general SO($2N$) gauge theories with complex scalar fields X in the adjoint representation (it is easy to generalize our results also to real scalar fields, or to fields of higher spin, including the gauge bosons). The scalar fields are $2N\times 2N$ complex antisymmetric matrices. For simplicity we will focus on operators involving only a single complex scalar field X. The gauge invariant operators

$$\mathrm{tr}(X^L) = X_{i_1 i_2} X_{i_2 i_3} X_{i_3 i_4}\cdots X_{i_L i_1}\,, \tag{4.2}$$

for L even and independent of N in the large-N limit, correspond in the 't Hooft large-N expansion to closed string vertex operators (in the AdS/CFT correspondence they map to supergravity states with angular momentum L on RP^5). The pfaffian operator is defined as:

$$\mathrm{Pf}(X) = \frac{1}{(2N)!}\epsilon^{i_1 i_2\cdots i_{2N}} X_{i_1 i_2} X_{i_3 i_4}\cdots X_{i_{2N-1} i_{2N}}\,. \tag{4.3}$$

The 't Hooft large-N expansion (in which $\lambda = 2N g_{YM}^2$ is held fixed when $N\to\infty$) of correlators with only single trace operators is well known [5, 22], and we will just review it briefly here. Feynman diagrams can be written in the double line ("fatgraph") notation, in which the propagator $\langle X_{ij} X^\dagger_{kl}\rangle$ is represented by two lines, each having one fundamental index. The propagator is given by:

$$\langle X_{ij}(x_1) X^\dagger_{kl}(x_2)\rangle \propto \frac{\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}}{(x_1-x_2)^2}\,. \tag{4.4}$$

We see that the indices are the same on both ends of the propagator but the existence of the two terms means that the direction of the lines in the Feynman diagram is not

[7] Closed surfaces will also appear, but they describe processes in which there is no interaction between the closed strings and the D-brane.

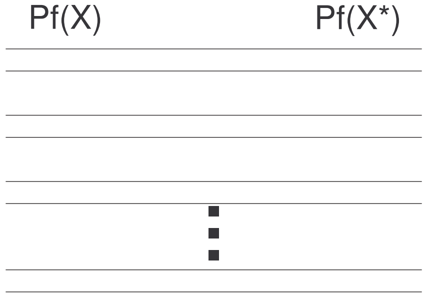

Figure 1: The free diagram in the $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\rangle$ correlation function. Each pfaffian is made of N basic fields. The propagators are unoriented, and could involve an exchange of the two lines.

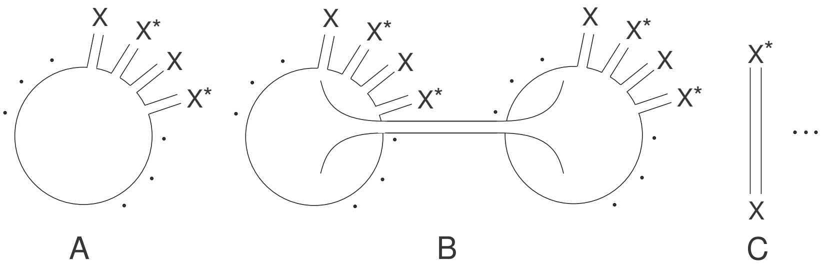

Figure 2: A general diagram for the $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\rangle$ correlation function. A connected diagram may have several connected components after removing the pfaffian vertices.

preserved. We will discuss theories (like supersymmetric Yang-Mills theories) whose interaction vertices are given by single traces, and there is always one index running on each closed loop. We will normalize the fields (obviously, the normalization does not affect normalized correlators) such that the whole action is proportional to $1/g_{YM}^2 = 2N/\lambda$, and then each propagator carries a factor of $1/2N$ and each interaction vertex a factor of $2N$ (ignoring the λ-dependence). Interpreting the Feynman diagram as a triangulation of a two dimensional surface, it then follows that the expansion in Feynman diagrams is an expansion in two dimensional compact closed topologies (both oriented and unoriented), where Feynman diagrams with each topology carry a power of $2N$ equal to their Euler characteristic (times some function of λ).

4.2 The $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\rangle$ correlation function

In this section we will analyze the form and the N-dependence of Feynman diagrams in the $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\rangle$ correlation function.

We will refer to $\mathrm{Pf}(X)$ as the "incoming pfaffian" and to $\mathrm{Pf}(X^\dagger)$ as the "outgoing pfaffian". The simplest diagram is the free diagram shown in figure 1, and it will be convenient to normalize other diagrams by dividing by it. In the $\mathcal{N}=4$ SYM theory this is in fact the only contribution to $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\rangle$, because the two-point function of chiral primary operators is not renormalized. However, our discussion here (until section 4.5) will apply also to non-supersymmetric field theories.

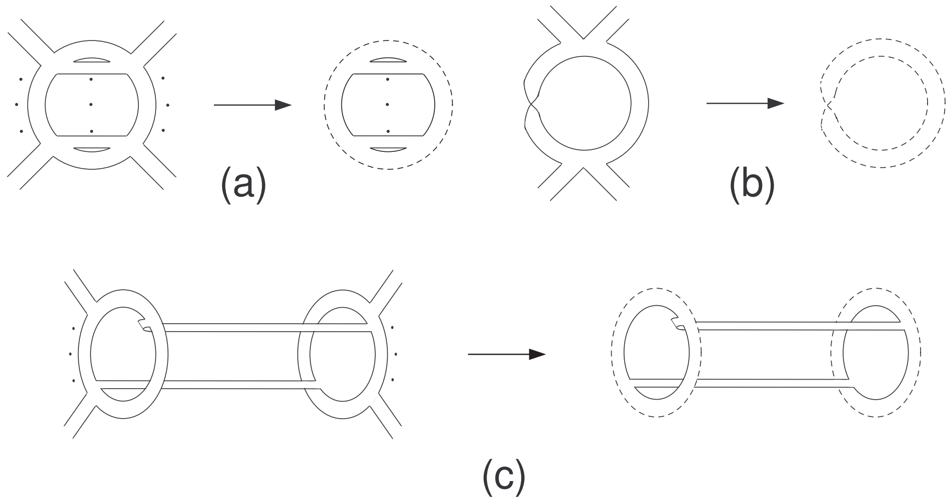

Figure 3: Amputating the diagram results in a vacuum diagram. A dashed line represents a boundary. (a) A disk diagram. (b) A Möbius strip. (c) A cylinder.

The general form of a Feynman diagram for this correlation function is shown in figure 2. There are N external legs originating from each of the pfaffians, which are arranged in several connected components in the diagram.[8] The simplest connected component is just a single propagator between an X and an $X^\dagger$ and we will refer to it occasionally as a "trivial component". Each single line from an external leg must connect to another external line. However, in the SO($2N$) theory (unlike the SU(N) case which will be discussed in the next section) no two indices on the incoming (or outgoing) pfaffian can be equal, so every single line coming from an external leg must go between the two pfaffians. This implies that the external legs are arranged in each of the connected components in an alternating order, as in figure 2, and that each component must include an equal number of legs from the incoming and outgoing pfaffians.

Suppose that we amputate the lines coming from external legs in any Feynman diagram (as shown in figure 3). The remaining legs, vertices and faces form a vacuum diagram, of the form considered in [5, 22], in which each amputated line becomes (part of) a boundary of the surface.[9] Each component of the diagram gives rise in this way to a (not necessarily oriented) compact surface. The amputation of a trivial component will be taken to be a disk. For instance, in the example of figure 2, the connected component A has the topology of a disk, and B has the topology of an annulus (we can create higher genus surfaces by adding handles and crosscaps). We will call these bounded surfaces the internal surfaces of the components.

[8]All the diagrams we discuss here will be connected, but we will separate the diagrams into components which are connected or disconnected when we remove the pfaffian vertices. All external legs from each pfaffian come from the same spacetime point, but it is more instructive to ignore this fact in the drawings.

[9]We will not really be relating these amputated components to vacuum diagrams, since these are different diagrams with different numerical values, but this is a convenient way of thinking about the N-dependence.

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This construction is motivated by the N-dependence of the diagrams, which is:

1. *External legs*: each external leg contributes a factor of $\frac{1}{2N}$ from its propagator, so the $2N$ external legs give $(2N)^{-2N}$. Note that although in every trivial component the two external legs are really one, our convention for the internal topology of such a component (a disk) will make up for the missing factor of $2N$ (see item 2).

2. *The topology of the internal surfaces*: this contribution is calculated as in the usual large-N expansion [5, 22]. The contribution of an internal surface is $(2N)^{\chi}$, where χ is the Euler characteristic of the internal surface, including the contribution of the boundaries. Here we do not sum over the external indices running on the boundaries of the surface (they do not contribute N factors). We will deal with these next.

3. *Combinatoric factors*: the number of possible ways of connecting external indices to the internal diagrams. This comes from the possible indices of the external legs, the number of equivalent diagrams arising from permutations of the external legs, and possible N-dependent symmetry factors.

The general picture that we will develop is that the internal topologies as they appear in these diagrams become the topologies of string theory. To do this we will need to verify that the topologies come with the correct power of $1/N$, and we will do so in section 4.4 (at leading order in $1/N$).

To determine the N-dependence, we need to evaluate the contribution of all three sources above. The first two are straightforward but the third will require some consideration. As a warm-up exercise, we will begin with the N counting of the free diagram, shown in figure 1, considering each of the three sources as follows:

1. The external leg propagators give $(2N)^{-2N}$.

2. The N trivial components count as N disks with $\chi = 1$, so we get $(2N)^{N}$.

3. We will compute the combinatoric factor in a way that is cumbersome for the free diagram, but will be more useful later. First we have two factors of $\frac{1}{(2N)!}$ from the definition of the pfaffian (2.1). Both the incoming and outgoing pfaffian are made out of N fields in a symmetric combination, hence there are $(N!)^2$ ways of choosing the order of the X's and the $X^{\dagger}$'s in figure 1. Since choosing the same pairs in a different order results in the same diagram, we counted every diagram $N!$ times, so we should divide by this symmetry factor. For a given pair we can still swap the two lines of the X and the two lines of the $X^{\dagger}$. This contributes a factor of 4 for every propagator. Again we counted each diagram several times so we should divide by a symmetry factor. If we swap the lines at both the incoming and outgoing ends of a propagator, we get the same diagram. The symmetry factor here is 2 for each propagator. After we chose how to connect the incoming fields to the outgoing fields, we sum over all possible arrangements of indices in the diagram. This gives us a factor of $(2N)!$ since we need to give each of the $2N$ external lines a different color index.

The N-dependence of the free diagram is therefore:

$$\text{free diagram} \propto \frac{(2\mathrm{N})!}{((2\mathrm{N})!)^2}\frac{(2^{\mathrm{N}}\mathrm{N}!)^2}{2^{\mathrm{N}}\mathrm{N}!}(2\mathrm{N})^{-\mathrm{N}} = \frac{\mathrm{N}!}{(2\mathrm{N})!}\mathrm{N}^{-\mathrm{N}}\,. \tag{4.5}$$

Proceeding to the general diagram, we consider a diagram with some number of connected components with different topologies, as in figure 2. The contribution of the three sources is:

1. Again, this is just $(2N)^{-2N}$.

2. From the internal topologies we get the factor $(2N)^{\sum\chi}$ where $\sum\chi$ is the sum of the Euler characteristics of all the internal surfaces (including the trivial disks).

3. We have, as before, two factors of $\frac{1}{(2N)!}$ coming from the definition (2.1). Now, we can change the order of all the N legs of each of the two pfaffians, and we can swap the two lines of each external leg since there is no orientation for the lines. These operations will not change the value of the diagram and they add a factor of $(2^N N!)^2$. Since all external lines start on the left and reach the right we have $(2N)!$ different ways for assigning values to the indices of the external legs. However, as for the free diagram, some diagrams were overcounted when we calculated the combinatoric factor. This happens when the diagram has some symmetry. This can happen in two ways. First, whenever the diagram has several identical components, some of the diagrams obtained by reordering the external legs give the same diagram but were counted as different diagrams. Each set of k identical components gives a symmetry factor of $k!$. Second, in every component which has just two external legs (one incoming and one outgoing), swapping the external legs of both gives back the same diagram, so there's a symmetry factor of 2^F, for a diagram with F two-legged components.[10]

   Putting all the factors together we get:

$$\frac{(2^N N!)^2}{\text{Symmetry Factors}}\frac{(2N)!}{((2N)!)^2}(2N)^{-2N}(2N)^{\sum\chi}\,, \tag{4.6}$$

   where the symmetry factors include the two contributions described above. After normalizing (for convenience) by dividing by the free diagram we find that the N-dependence of a general diagram is:

$$\frac{N!}{N^N}(2N)^{\sum\chi}\frac{1}{\text{Symmetry Factors}}\,. \tag{4.7}$$

   Note that the above analysis made use only of the color structure of the theory, and not of any special features of the theory, such as supersymmetry.

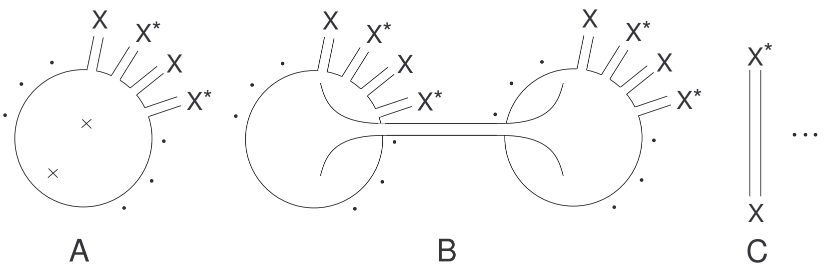

Figure 4: A typical diagram for $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\,\mathrm{tr}(X^J)\,\mathrm{tr}({X^\dagger}^J)\rangle$. The 'x' stands for insertions. When there is a U(1) symmetry acting on X, as in SYM theory, the two insertions must be in the same component, but this need not be the case in general.

4.3 Correlation functions with traces

Consider now correlators of the form

$$\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger) N\,\mathrm{tr}(X^{J_1}) N\,\mathrm{tr}(X^{J_2}) \cdots N\,\mathrm{tr}({X^\dagger}^{L_1}) N\,\mathrm{tr}({X^\dagger}^{L_2}) \cdots \rangle\,, \tag{4.8}$$

which in string theory are related to scattering of type-IIB supergravity particles off the D-brane. The general field theory diagram for this correlator will include some components which include trace insertions (in the interior of the surface as usual) and some that do not (figure 4 shows an example in which only one component has insertions, but of course this need not be the case in general).

The insertions function as ordinary interaction vertices in the theory as far as N-dependence is concerned (when we take each trace to come with a factor of N as above). The N dependence of these diagrams is therefore identical to that of the same diagram in $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\rangle$, given (after normalization) by (4.7).

4.4 Expansion in topologies

In this section we will derive the large-N expansion of the normalized correlators (4.1). As we discussed in section 3, in the expansion of the numerator of (4.1) on the string theory side, each surface in which the vertex operators are inserted appears together with all possible additions of disconnected empty (no vertex operators) surfaces. In the ratio (4.1), however, the normalization cancels the contribution of all 'empty' surfaces, and one is left with an expansion in N^χ where χ is the Euler characteristic of the topology of the surfaces in which the vertex operators are inserted.

We saw that we can characterize the field theory diagrams by the topologies of the internal components, and that a subset of these components, which we will call the interacting component, contains the trace insertions (this terminology is somewhat misleading, since the interacting 'component' may consist of several disconnected components). It is

[10] Additional, accidental symmetries may be present when the internal components are made of special, symmetric diagrams, and should be corrected for. These are rare, however, when we go to large 't Hooft coupling and the diagrams become dense, so we will ignore them.

natural to identify the topology of the interacting component with the corresponding topology in the string theory expansion of (4.1) (again, these may be disconnected). There are many diagrams with the same topology for the interacting component: different number of external legs can be attached to it, and there may be other internal components with arbitrary topologies. For example, the disk amplitude in string theory will correspond to all diagrams for which there is a single interacting component with the topology of a disk.

In order to justify this identification we need to check that the (normalized) contribution of all diagrams whose interacting component has Euler characteristic χ is proportional to N^χ, as it is on the string theory side.

The perturbative expansion of string theory amplitudes exhibits also 'factorization' properties. For example, the 'two disks' contribution to the scattering of two closed strings off the D-brane (each disk here contains two vertex operator insertions, one for an incoming and one for an outgoing string) is just the product of the two separate 'one disk' contributions to the scattering of each closed string off the D-brane 'on its own'. If our identification is correct, we expect to see the same behavior on the field theory side, though it is far from obvious why this should be the case since we are not discussing disconnected diagrams on the field theory side.

We will start by writing down explicit expressions for both the numerator and the denominator of (4.1) as sums of diagrams. We will then evaluate these expressions using the saddle-point method, and get a result of the expected form for (4.1), their ratio. Finally, we will use this result to confirm the 'factorization' property explained above.

4.4.1 $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\rangle$

Let $D_{k,\chi}$ denote the sum of all connected diagrams (with no vertex operator insertions) with k incoming and k outgoing legs (arranged in an alternating order) such that their internal topology has Euler characteristic equal to χ, without including the N-dependent contributions to the diagrams (but including the space-time dependence which we suppress). For $k = 1$ we will divide $D_{1,\chi}$ by two, to simplify the notation below.

Now, consider the sum of all diagrams which have $L_{k,\chi}$ connected components with k incoming (and k outgoing) external legs and internal Euler characteristic χ. Normalized by the free diagram, it is given by:

$$\frac{N!}{N^N}\prod_{k,\chi}\frac{(2N)^{\chi L_{k,\chi}}D_{k,\chi}^{L_{k,\chi}}}{L_{k,\chi}!}\,. \tag{4.9}$$

The product extends over $k = 1,\ldots,N$, $\chi = 1,0,-1,\ldots$. It is easy to see that the factorial terms in the denominators give the right symmetry factors for each diagram in the sum. Recall that there is another symmetry factor of $2^{-L_{1,\chi}}$, which we take into account by our insertion of a $1/2$ in the definition of $D_{1,\chi}$.

To obtain the full correlation function $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\rangle$ we need to sum over all possible diagrams, which is the same as summing over all possibilities for $L_{k,\chi}$, and we get:

$$\sum_{L_{k,\chi}\geq 0}\frac{N!}{N^N}\prod_{k,\chi}\frac{(2N)^{\chi L_{k,\chi}}D_{k,\chi}^{L_{k,\chi}}}{L_{k,\chi}!}\,, \tag{4.10}$$

where the sum is constrained by the condition $\sum_{k,\chi} kL_{k,\chi} = N$. We will denote the same sum (4.10) with the constraint $\sum_{k,\chi} kL_{k,\chi} = n$ by a_n.

4.4.2 $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\,\mathrm{tr}(X^{J_1})\,\mathrm{tr}(X^{J_2})\cdots\mathrm{tr}(X^{\dagger L_1})\,\mathrm{tr}(X^{\dagger L_2})\cdots\rangle$

Turning our attention to correlation functions with insertions, let us evaluate the contribution of all diagrams whose interacting component has Euler characteristic χ_0.[11] Every such diagram is characterized by the number k_0 of incoming (and outgoing) external legs connected to its interacting component, $1 \leq k_0 \leq N$, and, as before, by $L_{k,\chi}$, the number of (insertion free) components with Euler characteristic χ and k incoming (and outgoing) external legs. These satisfy $\sum_{k,\chi} kL_{k,\chi} + k_0 = N$.

Let $C_{k,\chi}$ denote the sum of all diagrams with k incoming (and k outgoing) legs (arranged in alternate order on each component) which include the trace insertions and such that their internal topology has Euler characteristic χ, again without including the N-dependence of the diagrams. We could also consider here just a specific topology with this Euler characteristic, and this will be useful in section 4.4.4, but it wouldn't affect the qualitative result.

The contribution of all diagrams that have specific values of k_0 and $L_{k,\chi}$ is

$$\frac{N!}{N^N}(2N)^{\chi_0} C_{k_0,\chi_0} \prod_{k,\chi} \frac{(2N)^{\chi L_{k,\chi}} D_{k,\chi}^{L_{k,\chi}}}{L_{k,\chi}!}\,. \tag{4.11}$$

The total correlation function comes from summing over all values of k_0 and $L_{k,\chi}$:

$$\sum_{L_{k,\chi};k_0>0} \frac{N!}{N^N}(2N)^{\chi_0} C_{k_0,\chi_0} \prod_{k,\chi} \frac{(2N)^{\chi L_{k,\chi}} D_{k,\chi}^{L_{k,\chi}}}{L_{k,\chi}!}\,, \tag{4.12}$$

where the sum is constrained by the condition $\sum_{k,\chi} kL_{k,\chi} + k_0 = N$. We will denote the same sum (4.12) with the constraint $\sum_{k,\chi} kL_{k,\chi} + k_0 = n$ by b_n.

4.4.3 Saddle point evaluation

We want to show that the leading N-dependence of the contribution of diagrams whose interacting component has Euler characteristic χ_0 to (4.1), which is the ratio of (4.12) and (4.10), is given by $(2N)^{\chi_0}$. We will evaluate it using the saddle point method, as follows.

Define the 'partition functions':

$$\begin{aligned} Z_1(\beta) &\equiv \sum_{n=0}^{\infty} a_n \beta^n \\ &= \frac{N!}{N^N} \sum_{\{L_{k,\chi}\geq 0\}} \beta^{\sum_{k\geq 1,\chi\leq 1} kL_{k,\chi}} \prod_{k,\chi} \frac{(2N)^{\chi L_{k,\chi}} D_{k,\chi}^{L_{k,\chi}}}{L_{k,\chi}!} \end{aligned}$$

[11]Note that the interacting 'component' may actually consist of several disconnected components, and then χ_0 could be as large as the number of disconnected components.

$$= \frac{N!}{N^N}\prod_{k,\chi}\sum_{\{L_{k,\chi}\geq 0\}}\frac{(\beta^k D_{k,\chi}(2N)^\chi)^{L_{k,\chi}}}{L_{k,\chi}!} = \frac{N!}{N^N}e^{h(\beta)}\,, \tag{4.13}$$

where $h(\beta) \equiv \sum_{k\geq 1,\chi\leq 1}\beta^k D_{k,\chi}(2N)^\chi$, and

$$\begin{aligned}
Z_2(\beta) &\equiv \sum_{n=0}^{\infty} b_n\beta^n \\
&= \frac{N!}{N^N}(2N)^{\chi_0}\sum_{\{L_{k,\chi}\geq 0,k_0>0\}}\beta^{k_0+\sum_{k\geq 1,\chi\leq 1}kL_{k,\chi}}C_{k_0,\chi_0}\prod_{k,\chi}\frac{(2N)^{\chi L_{k,\chi}}D_{k,\chi}^{L_{k,\chi}}}{L_{k,\chi}!} \\
&= (2N)^{\chi_0}\left(\sum_{k_0\geq 1}\beta^{k_0}C_{k_0,\chi_0}\right)\frac{N!}{N^N}\prod_{k,\chi}\sum_{\{L_{k,\chi}\geq 0\}}\frac{(\beta^k D_{k,\chi}(2N)^\chi)^{L_{k,\chi}}}{L_{k,\chi}!} \\
&= (2N)^{\chi_0}g_{\chi_0}(\beta)Z_1(\beta)\,,
\end{aligned} \tag{4.14}$$

where $g_{\chi_0}(\beta) \equiv \sum_{k\geq 1}\beta^k C_{k,\chi_0}$. We are interested in computing the correlation functions (4.10) $= a_N$ and (4.12) $= b_N$. These coefficients are given by the contour integrals:

$$\begin{aligned}
a_N &= \frac{1}{2\pi i}\oint Z_1(\beta)\beta^{-(N+1)}d\beta\,, \\
b_N &= \frac{1}{2\pi i}\oint Z_2(\beta)\beta^{-(N+1)}d\beta = (2N)^{\chi_0}\frac{1}{2\pi i}\oint g_{\chi_0}(\beta)Z_1(\beta)\beta^{-(N+1)}d\beta\,.
\end{aligned} \tag{4.15}$$

We will begin by analyzing a_N. Keeping only the two leading (first and zeroth) powers in N in $h(\beta)$ (the leading term cannot vanish since the theory has a propagator) we get

$$a_N \simeq \frac{1}{2\pi i}\frac{N!}{N^N}\oint e^{2Nf(\beta)}e^{f_0(\beta)}\frac{d\beta}{\beta}\,, \tag{4.16}$$

where we define $f(\beta) \equiv \sum_{k\geq 1}\beta^k D_{k,1} - \frac{1}{2}\log(\beta)$, including the leading terms in h, and $f_0(\beta) \equiv \sum_{k\geq 1}\beta^k D_{k,0}$, with the subleading terms; both functions are independent of N.

Since $N \gg 1$, we may use the saddle point method. In this method, one considers a contour along which the imaginary part of $f(\beta)$ is fixed (this is the trajectory of steepest descent for the real part of f). Such a contour, if it exists, will pass through some extremal points of $f(\beta)$ where $\mathrm{Re}(f(\beta))$ is maximal. Since N is large, the main contribution to the integral comes from the vicinity of those points, where we can approximate the integrand as a (very narrow) gaussian. The question is whether in fact there is such a contour with constant $\mathrm{Im}(f(\beta))$ which circles the origin, possibly going to infinity as long as $\mathrm{Re}(f(\beta)) \to -\infty$ there. We will conjecture that this is indeed the case.

Assume first, for simplicity, that there is one saddle point β_0, for which $f'(\beta_0) = 0$. We then get

$$a_N \simeq \frac{1}{2\pi i}\frac{N!}{N^N}e^{i\alpha}\frac{\sqrt{2\pi}e^{2N|f(\beta_0)|}e^{f_0(\beta_0)}}{\sqrt{2N|f''(\beta_0)|}\beta_0} \tag{4.17}$$

where $e^{i\alpha}$ is a phase determined by the direction of steepest descent.

As the simplest example, consider a theory in which only $D_{1,1} \neq 0$. The only diagram is the free one so we expect to get $a_N = (2D_{1,1})^N$ (recall that our expressions for the diagrams are normalized by the free diagram with a propagator equal to one and that $D_{1,1}$ was defined as half the propagator). Indeed, in this case we have $f(\beta) = D_{1,1}\beta - \frac{1}{2}\log(\beta)$ so $\beta_0 = 1/2D_{1,1}$ and so

$$a_N \simeq \frac{1}{2\pi i}\frac{N!}{N^N}e^{i\alpha}\frac{\sqrt{2\pi}e^{2N(\frac{1}{2}+\frac{1}{2}\log(2D_{1,1}))}}{\sqrt{2N2D_{1,1}^2(2D_{1,1})^{-1}}} = -ie^{i\alpha}(2D_{1,1})^N\frac{N!e^N}{N^N\sqrt{2\pi N}} = (2D_{1,1})^N \quad (4.18)$$

where in the last equality we used the Stirling approximation. In this example we can explicitly verify that a contour of 'steepest descent' encircling the origin in fact exists, justifying the saddle point evaluation, and that the phases cancel out properly.

We can make a similar analysis for b_N. The only difference is the appearance of another (N-independent) function $g_{\chi_0}(\beta)$ in the integrand alongside the steep exponential. The saddle point method applies here as well, and since the gaussian is very narrow, one can take $g_{\chi_0}(\beta)$ as a constant equal to $g_{\chi_0}(\beta_0)$. Aside from this factor, the result is the same (since the saddle point is the same), so we get:

$$b_N \simeq \frac{1}{2\pi i}\frac{N!}{N^N}(2N)^{\chi_0}e^{i\alpha}\frac{\sqrt{2\pi}g_{\chi_0}(\beta_0)e^{2N|f(\beta_0)|}e^{f_0(\beta_0)}}{\sqrt{2N|f''(\beta_0)|}\beta_0}\,. \quad (4.19)$$

Most terms cancel out when we take the ratio

$$\frac{b_N}{a_N} \simeq g_{\chi_0}(\beta_0)(2N)^{\chi_0}\,, \quad (4.20)$$

which is indeed proportional to $(2N)^{\chi_0}$ as expected.

When there is more than one saddle point along the integration contour things don't cancel out so neatly. However, if there is one saddle point for which $\mathrm{Re}(f(\beta))$ is the largest, its contribution to the integral strongly dominates in both a_N and b_N, so we again get the same result. We run into trouble when there are two or more saddle points for which $\mathrm{Re}(f(\beta))$ is the same. $\mathrm{Re}(f(\beta))$ is a function of β with power series coefficients that depend on the 't Hooft coupling λ. It would be surprising if this function would have the property that it has the same value at two extremal points for all values of λ, so we may expect our result (4.20) to hold at least for generic values of the 't Hooft coupling.

To summarize, up to an assumption about the existence of an appropriate contour of constant phase encircling the origin, we have shown that the correlation functions behave as we claimed, with the leading contribution of diagrams whose interacting component has Euler characteristic χ_0 coming with a coefficient of N^{χ_0} as in string theory.

4.4.4 Factorization

There is another test we can make of our identification. In string theory, the scattering of two string states $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger) N\,\mathrm{tr}(X^l) N\,\mathrm{tr}(X^{\dagger l}) N\,\mathrm{tr}(X^m) N\,\mathrm{tr}(X^{\dagger m})\rangle$ gets a contribution from the disconnected topology having two components, with Euler characteristics χ_1 and χ_2, the first of which contains the $\{\mathrm{tr}(X^l), \mathrm{tr}(X^{\dagger l})\}$ insertions and the second of which contains the $\{\mathrm{tr}(X^m), \mathrm{tr}(X^{\dagger m})\}$ insertions. This contribution is the product of the χ_1 contribution to $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger) N\,\mathrm{tr}(X^l) N\,\mathrm{tr}(X^{\dagger l})\rangle$ and the χ_2 contribution to $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger) N\,\mathrm{tr}(X^m) N\,\mathrm{tr}(X^{\dagger m})\rangle$ (all divided by $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\rangle$).

The same equality should hold in the field theory. Note that in section 4.4.2 we did not need to assume that the interacting component was connected. According to (4.20), we should check that

$$g_{\chi_1\otimes\chi_2}(\beta_0)(2N)^{\chi_1+\chi_2} = g_{\chi_1}(\beta_0) g_{\chi_2}(\beta_0)(2N)^{\chi_1}(2N)^{\chi_2}\,. \tag{4.21}$$

But $g_\chi(\beta) = \sum_{k\geq 1}\beta^k C_{k,\chi}$. Recall that $C_{k,\chi_1\otimes\chi_2}$ is the sum of all diagrams (without the N-dependence) with k external legs and internal topology consisting of two components, one with topology χ_1 and the other χ_2. The k external legs will be divided between these two components, so $C_{k,\chi_1\otimes\chi_2} = \sum_{k_1+k_2=k} C_{k_1,\chi_1} C_{k_2,\chi_2}$. It then follows that $\sum_{k_1,k_2}\beta^{k_1}\beta^{k_2} C_{k_1,\chi_1} C_{k_2,\chi_2} = \sum_k \beta^k C_{k,\chi_1\otimes\chi_2}$ or in other words, that we have $g_{\chi_1\otimes\chi_2}(\beta) = g_{\chi_1}(\beta) g_{\chi_2}(\beta)$, which confirms (4.21).

It is important to note that the above analysis is applicable to any theory with the same 'color structure'. We made no use of the special properties (most notably, the supersymmetry) of the $\mathcal{N}=4$ SYM theory, which has a known equivalent string theory in which we know that the pfaffian operator should actually correspond to a D-brane.

4.5 The supersymmetric theory

In this section we discuss the $\mathcal{N}=4$ SYM theory with SO$(2N)$ gauge group. This theory contains three complex scalar fields X in the adjoint representation, transforming under the SO(6) R-symmetry, and we can discuss operators as above made from one of these scalar fields. This field theory is dual to type-IIB string theory on $AdS_5\times RP^5$ [6].

In the simple example above, (4.18), we saw that if only $D_{1,1}\neq 0$ our approximation gives the right result, i.e. the free diagram only. On the other hand, in the $\mathcal{N}=4$ SYM theory $\mathrm{Pf}(X)$ is a chiral primary so $\langle \mathrm{Pf}(X)\,\mathrm{Pf}(X^\dagger)\rangle$ is protected. All diagrams cancel out except for the free diagram, so $a_N = (2D_{1,1})^N$ exactly. This could be taken to suggest that in this theory all the $D_{k,\chi}$'s vanish except for the propagator $D_{1,1}$ (which is independent of the 't Hooft coupling in this case). If this is true then in this theory we know for sure that the saddle point computation is justified, and that $\beta_0 = 1/2D_{1,1}$. However, we have not been able to prove this.

Assuming that we are studying a supersymmetric theory in which this is indeed correct, there is another calculation we can make. The contribution of diagrams in which the interacting component has l incoming external legs is proportional to (taking the propagator to be one for simplicity):

$$\frac{N!}{N^N}(2N)^\chi \frac{N^{N-l}}{(N-l)!}\,, \tag{4.22}$$

because (with our assumption) the only contribution comes from having trivial propagators in the rest of the diagram and these give this symmetry factor.

It is easy to see that diagrams with larger values of l are suppressed in the large-N limit, so that diagrams whose number of trivial legs is of order N dominate. In fact, from (4.22), given the topology, the dependence of the diagram on l is proportional to $\frac{N(N-1)\cdots(N-l+1)}{N^l}$ (ignoring the N-independent factor $C_{l,\chi}$). We have

$$\begin{aligned} \log\left(\frac{N(N-1)\cdots(N-l+1)}{N^l}\right) &= \sum_{j=1}^{l} \log\left(1-\frac{j-1}{N}\right) \\ &= -\frac{\sum_{j=0}^{l-1} j}{N} + O\left(\frac{l^3}{N^2}\right) \\ &= -\frac{l(l-1)}{2N} + O\left(\frac{l^3}{N^2}\right), \end{aligned} \tag{4.23}$$

so the dependence of the N-factor of the diagram on l is proportional to the factor $Exp(-\frac{l(l-1)}{2N})(1+O(\frac{l^3}{N^2}))$ (when $l \approx N$ it is easy to see that we get $\approx Exp(-N)$). This means that it is sufficient to consider diagrams in which the number of legs attached to the internal surface is up to a number of order $\sqrt{N}$, and this is also the expectation value of the number of the interacting legs.

The internal surface represents the worldsheet in string theory, and we may think of its boundary as a closed string boundary state. The internal surface is really part of a discrete Feynman diagram and not a continuous surface like the worldsheet, so the boundary state is represented in the CFT as made from a number of discrete units. There are roughly the same number of vertices on the boundary as there are external legs, which means that the boundary state string has $\sim \sqrt{N}$ discrete units. This is somewhat similar to [10] where closed string states in the plane wave limit of $AdS_5 \times S^5$ correspond to traces of $\sim \sqrt{N}$ fields in the field theory, so they are also made of $\sim \sqrt{N}$ discrete units.

5. SU(N) gauge theories

In this section we will deal with SU(N) gauge theories. In these theories, (3.4) takes the form

$$\frac{\langle \det_L(X)\det_L(X^\dagger) N\,\mathrm{tr}(X^{l_1}) N\,\mathrm{tr}(X^{l_2})\cdots N\,\mathrm{tr}(X^{\dagger m_1}) N\,\mathrm{tr}(X^{\dagger m_2})\cdots\rangle}{\langle \det_L(X)\det_L(X^\dagger)\rangle}. \tag{5.1}$$

As discussed in section 3, we expect (5.1) to have a topological large-N expansion which will include all oriented surfaces with boundaries (at least for the $\mathcal{N}=4$ SYM theory, which is equivalent to a string theory), and we will develop this expansion in this section. The analysis will basically parallel that of the SO($2N$) case (section 4) with a few changes.

The outline of this section is as follows: in section 5.1 we will review the usual large-N expansion of this theory, which applies only to correlators of traces of small powers of X. In section 5.2 we will analyze the form and N-dependence of a general diagram in $\langle \det_L(X)\det_L(X^\dagger)\rangle$, and in section 5.3 we will extend the analysis to diagrams for correlators which also include trace operators. We will identify the topology corresponding to every diagram and find that all oriented topologies, including those with boundaries, appear.

The analysis will make use only of the general color structure of the theory. The main difference from the SO($2N$) case is in the different constraints on the indices in the definitions of $\det_L(X)$. One of the results will be the appearance of a class of diagrams which do not exist for SO($2N$), and it will be natural to interpret them as processes with intermediate brane states of angular momentum different from L, such as $\det_M(X)$, $M \neq L$ (similar subleading diagrams do not exist for pfaffians in the SO($2N$) theory and this is consistent with the fact that there are no pfaffian-like states with angular momentum less than N). Note that we could also analyze operators of the type $\det_L(X)$ in the SO($2N$) gauge theory, and we would find similar results to the ones we present here.

In section 5.4 we will discuss the large-N expansion of (5.1). We will argue that the contribution of all diagrams with a specific topology is proportional to N^χ, where χ is the Euler characteristic of that topology. This will justify our identification of the topology of each Feynman diagram. The discussion of factorization here is completely identical to the SO($2N$) case, so we will not repeat it.

5.1 Review and definitions

Let X be a scalar field in the adjoint representation of SU(N) (the $\mathcal{N} = 4$ SYM theory contains three such complex R-charged fields). We may label X by one fundamental and one anti-fundamental index, X_i^j, where $i, j = 1, \ldots, N$. In this notation, the operators of interest to us are the traces

$$\mathrm{tr}(X^L) = X_{i_2}^{i_1} X_{i_3}^{i_2} X_{i_4}^{i_3} \cdots X_{i_1}^{i_L} \tag{5.2}$$

with $L \ll N$, and the subdeterminant operators

$$\det{}_L(X) = \frac{1}{L!(N-L)!} \epsilon_{i_1 \cdots i_L i_{L+1} \cdots i_N} \epsilon^{j_1 \cdots j_L i_{L+1} \cdots i_N} X_{j_1}^{i_1} \cdots X_{j_L}^{i_L} \tag{5.3}$$

with $N - L \ll N$. On the string theory side, the operators (5.2) correspond to closed string states, gravitons and their superpartners, with small angular momentum L ($\ll N$) on S^5, and the operators (5.3) correspond to "giant gravitons", states with large angular momentum L ($N - L \ll N$) on S^5, which become spherical D3-branes. As explained in section 3, based on string perturbation theory we expect correlators of closed string states to have a large-N topological expansion with only even powers of $1/N$ (corresponding to oriented closed surfaces), while correlators of closed string states with "giant gravitons" should have a large-N topological expansion with even as well as odd powers of $1/N$ (corresponding to oriented surfaces with boundaries).

The 't Hooft large-N expansion (in which $\lambda = N g_{YM}^2$ is held fixed) of correlators of usual single-trace operators is well known [5] and we will briefly review the essential points.

The Feynman diagrams of an SU(N) gauge theory with adjoint fields can be written in the double line notation, in which the propagator $\langle X_j^i X_l^{\dagger k} \rangle$ is represented by two directed lines, one with the fundamental index and one with the anti-fundamental index (figure 5(a)). The propagator is given by

$$\langle X_j^i(x_1) X_l^{\dagger k}(x_2) \rangle \propto \frac{\delta_l^i \delta_j^k - \frac{1}{N} \delta_j^i \delta_l^k}{(x_1 - x_2)^2}, \tag{5.4}$$

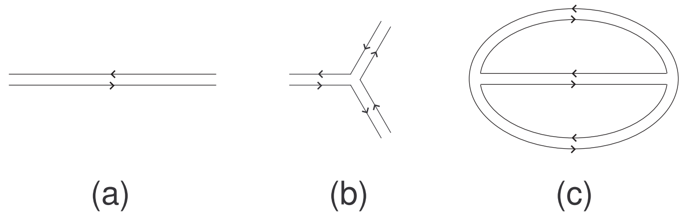

Figure 5: The double line notation: (a) the propagator (b) a vertex (c) the free planar diagram for $\langle \mathrm{tr}(X^3)\,\mathrm{tr}(X^{\dagger 3})\rangle$.

so to leading order in $1/N$, the incoming and outgoing indices on each line in a propagator are equal. The interaction terms that appear in the lagrangian are single traces (see figure 5(b)). When a closed loop is formed in the diagram the index runs over all possible values and contributes a factor of N to the value of the diagram. It is convenient to normalize the fields such that there is a factor of $1/g_{YM}^2 = N/\lambda$ in front of the whole lagrangian and no other dependence on g_{YM}. Then, each vertex in a Feynman diagram gives a factor of N/λ and each propagator gives a factor of λ/N.

Thus, the N dependence of a vacuum Feynman diagram in the large-N limit with fixed λ is, to leading order in $1/N$, N^{V-E+F}, where V is the number of vertices, E the number of propagators, and F the number of closed loops in the diagram. We may think of the diagram as a triangulation of a two-dimensional surface, with a face corresponding to each closed loop, and an edge to each (double lined) propagator. The leading N power of the diagram, $V - E + F$, is then equal to the Euler characteristic of the surface. A topological theorem states that every compact two-dimensional oriented surface is homeomorphic to a sphere with a certain number of handles and boundaries (holes), and that all triangulations of the surface have the same value of $V - E + F$, given by $\chi = 2 - 2H - B$ where H is the number of handles and B is the number of boundaries.

Incoming and outgoing graviton states correspond to trace operators similar to those appearing as interaction vertices. Hence, the power of $1/N$ corresponding to a diagram involving such states can be calculated in the same way as for vacuum diagrams, replacing the graviton insertions by vertices. All the surfaces constructed in this way are closed, i.e. without boundaries, so their Euler characteristics are even. Thus, the large-N expansion only includes even powers of N. The leading diagrams will correspond to spheres (which means they are planar), followed by the torus and so on.

5.2 The $\langle \det_L(X)\,\det_L(X^\dagger)\rangle$ correlation function

We will begin with $\langle \det_L(X)\,\det_L(X^\dagger)\rangle$ with no traces in the correlator. The general form of a Feynman diagram for this correlator is shown in figure 6. Using similar terms to the SO($2N$) case, it consists of several connected components, each having a number of external legs connected to it, but in this theory they are attached in an arbitrary order. Unlike the SO($2N$) case, the external legs do not have to alternate between the incoming and outgoing determinants.

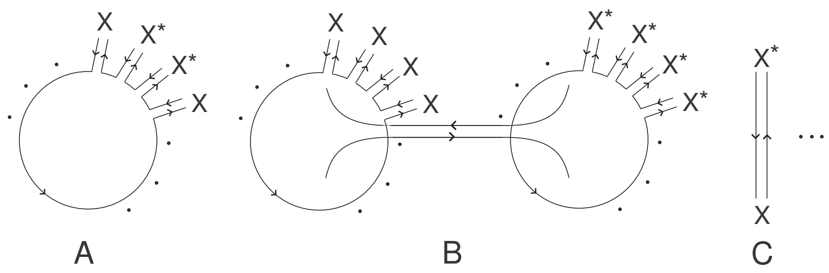

Figure 6: The general form of a Feynman diagram for $\langle \det_L(X) \det_L(X^\dagger)\rangle$.

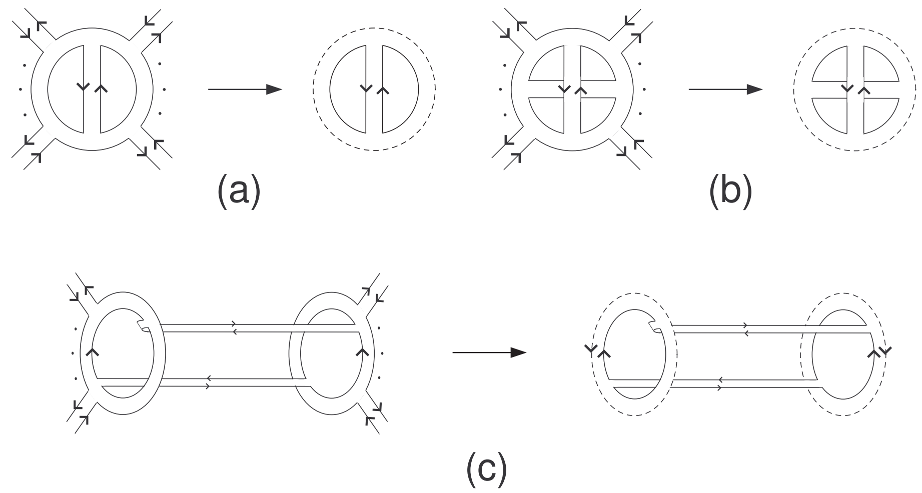

Figure 7: Amputating the diagrams. Boundaries are denoted by dashed lines. (a) A disk diagram. (b) A torus with one hole. (c) A cylinder (two holes in one connected component).

We begin by developing some terminology. First, just as in the SO$(2N)$ theory, imagine that we amputate the lines coming from external legs. The remaining edges and vertices form a vacuum diagram which contains only trace vertices (figure 7). The diagram forms a surface which can be treated using the usual large-N expansion. As in the SO$(2N)$ case, we will regard each amputated line as (part of) a boundary for this surface, and we will call these surfaces the internal surfaces of the diagram. As in section 4, we will consider a trivial component (one propagator running from X to $X^\dagger$) to have the internal topology of a disk.

Second, we have already noted that in this theory, the external legs are attached to the internal surfaces in an arbitrary order, unlike the SO$(2N)$ case where there was only one way to connect them because of the antisymmetric nature of the pfaffian operators. In the double line notation, each propagator is represented by two lines directed in opposite directions and carrying one index each. Since there are L incoming external legs and L outgoing external legs in a diagram, there are $2L$ directed external lines in the diagram. Those lines can be of one of four types: they either go from X to $X^\dagger$, from X to X, from $X^\dagger$ to X or from $X^\dagger$ to $X^\dagger$. Let $L_{XX^\dagger}$, L_{XX}, $L_{X^\dagger X}$, and $L_{X^\dagger X^\dagger}$ denote the number of lines (and hence indices) belonging to each type, respectively. It is clear (see figure 8) that $L_{XX} = L_{X^\dagger X^\dagger}$ and $L_{XX^\dagger} = L_{X^\dagger X}$.

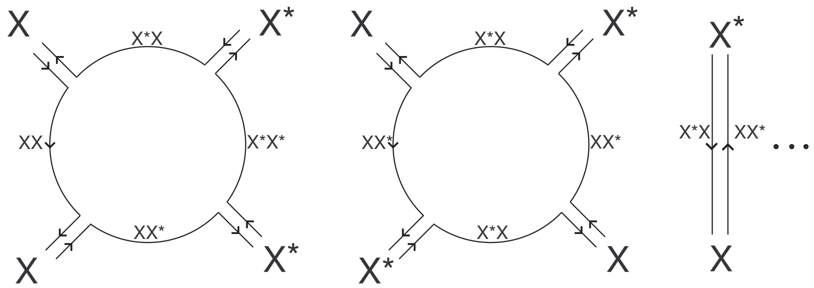

Figure 8: Types of external lines. The lines can be divided into four types according to their origin and destination.

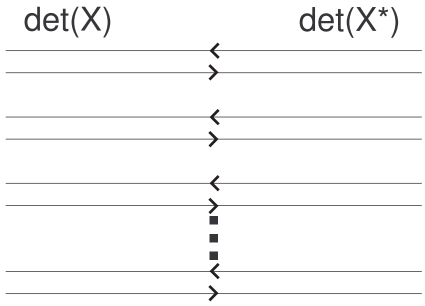

Figure 9: The free diagram for $\langle \det_L(X) \det_L(X^\dagger) \rangle$. The difference from the SO$(2N)$ case is that the propagators have direction.

What are the possible values of $L_{XX^\dagger}$? Each trivial component in the diagram contributes one to $L_{XX^\dagger}$. The other external propagators connect to the boundaries of the internal surfaces. There are several ways to attach them, each giving a different value of $L_{XX^\dagger}$. When the external legs are arranged in alternating order (one from $\det_L(X)$, one from $\det_L(X^\dagger)$, and so on) $L_{XX^\dagger}$ will be equal to its maximal value, L. Each swap of neighboring external legs reduces $L_{XX^\dagger}$ by one.

Consider a diagram with $L_{XX^\dagger} = L$. In such a diagram, each cut in the diagram (between the $\det_L(X)$ and $\det_L(X^\dagger)$ vertices) will cross at least L pairs of (single index) external lines. In a diagram with $L_{XX^\dagger} = L - 1$, however, it is possible to cut the diagram in such a way that the cut will cross only $L - 1$ pairs of external lines. When $L_{XX^\dagger}$ decreases further, there are cuts which cross smaller numbers of line pairs. It is natural to interpret diagrams which have cuts crossing K pairs of external lines as diagrams which have intermediate states involving $\det_K(X)$, or in string theory language, processes which have "giant gravitons" with a lower angular momentum K as intermediate states. The fact that similar diagrams do not exist for pfaffians in the SO$(2N)$ theory is consistent with the fact that there are no pfaffian-like states with angular momentum less than N.

In any case, we will see that the above characteristics of a general diagram, the topology of the internal surfaces and the value of $L_{XX^\dagger}$, fully determine its N-dependence.

As a simple example, we start with the calculation of the N dependence of the free diagram shown in figure 9. First we note that the $\frac{1}{L!}$ factor in the definition of $\det_L(X)$ can be cancelled by all the different permutations of the incoming external legs. The same is true for $\det_L(X^\dagger)$ and permutations of outgoing external legs. However, when we permute the legs of both X and $X^\dagger$ there are $L!$ permutations which give the same diagram. Since we want to count each diagram only once we need to divide by this symmetry factor.

Every line has an index which is summed over, but the indices are constrained by the definition (2.2) in a complicated way. We therefore need to count the different ways to assign indices to every line. Since the lines are directed we can divide them into two types. Lines starting on the incoming subdeterminant and ending on the outgoing subdeterminant we call the $XX^\dagger$ type, and lines starting on the outgoing subdeterminant and ending on the incoming subdeterminant we call the $X^\dagger X$ type (as shown in figure 8). The definition of the subdeterminant implies that the group of the L indices of type $XX^\dagger$ and the group of the L indices of type $X^\dagger X$ must be the same set of different numbers. To choose such a set there are $\frac{N!}{(N-L)!L!}$ ways, and there are $(L!)^2$ ways of assigning them to the different lines. The L propagators give N^{-L}. Putting it all together we get

$$\frac{1}{L!}\frac{N!}{(N-L)!L!}L!L!N^{-L} = \frac{N!}{(N-L)!}N^{-L}. \tag{5.5}$$

We now proceed to the calculation of a general diagram. As in the SO($2N$) theory, there are three sources for the N dependence of a diagram. The analysis of the first two parts is identical to the SO($2N$) theory, but the third part will be different.

1. External propagators: there are $2L$ external propagators and they give a factor of N^{-2L}. Also, the $1/L!$ factors in the definitions of $\det_L(X)$ cancel the various permutations of the external legs, up to whatever symmetry factor the diagram has: every group of C identical components in a diagram gives a $1/C!$ symmetry factor.

2. Internal surfaces: the edges, vertices and loop indices of the internal surfaces contribute N^χ each, where χ is the Euler characteristic of the internal topology (recall that we consider each trivial component to be a disk). The definition of the boundaries of the surfaces means that we have not taken into account the counting of the indices which run on these boundaries. We will do so now.

3. Counting the external indices: the count depends on how the external legs connect to the boundaries of the internal surfaces. The definition (5.3) implies that all incoming lines must carry different indices, and the same for the outgoing lines, and that these two index sets must be the same. For the general diagram there are $\frac{N!}{(N-L_{XX^\dagger})!}$ ways to choose the $XX^\dagger$ indices. The XX indices must all be different from these, so we can choose them in $\frac{(N-L_{XX^\dagger})!}{(N-L)!}$ different ways. The same applies to the $X^\dagger X^\dagger$ indices, which are independent of the XX indices, and there are $\frac{(N-L_{XX^\dagger})!}{(N-L)!}$ ways to choose them as well. We still have to choose the $X^\dagger X$ indices. Since the incoming and outgoing indices on each side must be the same set, and the $X^\dagger X$ indices must be different from both the XX indices and the $X^\dagger X^\dagger$ indices, it follows that the set

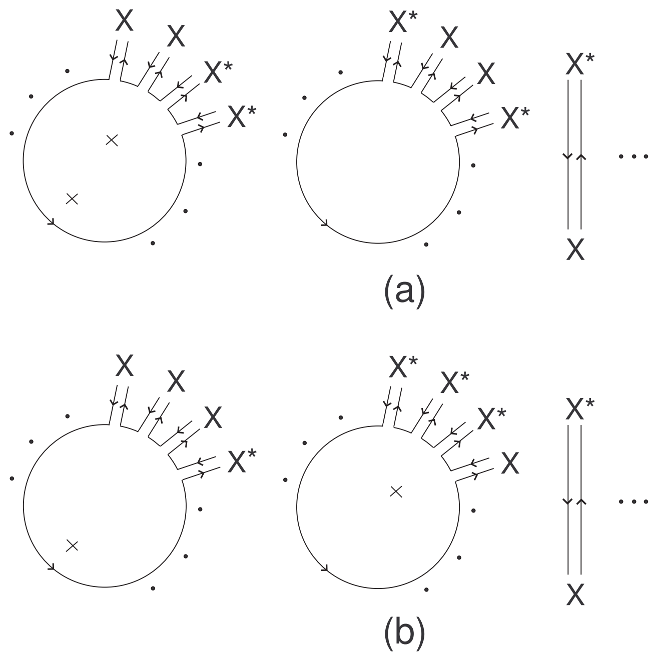

Figure 10: Some possible diagrams appearing in the computation of the correlation function $\langle \det_L(X) \det_L(X^\dagger) N \operatorname{tr}(X^l) N \operatorname{tr}(X^{\dagger l}) \rangle$.

of $X^\dagger X$ indices must equal the set of $XX^\dagger$ indices. This leaves us with just $L_{XX^\dagger}!$ possible permutations. Altogether we get

$$\frac{N!}{(N - L_{XX^\dagger})!} \left(\frac{(N - L_{XX^\dagger})!}{(N-L)!} \right)^2 L_{XX^\dagger}! = \frac{N!(N - L_{XX^\dagger})! L_{XX^\dagger}!}{(N-L)!^2} . \tag{5.6}$$

Combining the three parts together, we get

$$\frac{1}{\text{Symmetry Factor}} N^{-2L} N^{\chi} \frac{N!(N - L_{XX^\dagger})! L_{XX^\dagger}!}{(N-L)!^2} . \tag{5.7}$$

It will be convenient to normalize by dividing by the free diagram (5.5). This gives

$$\frac{N^{-L}}{(N-L)!} \frac{1}{\text{Symmetry Factors}} (N - L_{XX^\dagger})! L_{XX^\dagger}! N^{\sum \chi} , \tag{5.8}$$

where $\sum \chi$ is the sum of the Euler characteristics of the internal surfaces.

5.3 Correlation functions with traces

Consider the correlator

$$\langle \det_L(X) \det_L(X^\dagger) N \operatorname{tr}(X^{l_1}) N \operatorname{tr}(X^{l_2}) \cdots N \operatorname{tr}(X^{\dagger m_1}) N \operatorname{tr}(X^{\dagger m_2}) \cdots \rangle . \tag{5.9}$$

We can think of the single-trace operators as internal interaction vertices appearing in the diagram rather than external incoming or outgoing states. The general diagram for this correlator will therefore look something like those in figure 10. Just like the diagrams in $\langle \det_L(X) \det_L(X^\dagger)\rangle$, it consists of several components, some of which contain trace insertions. The N dependence of these diagrams is identical to that of the same diagram in $\langle \det_L(X) \det_L(X^\dagger)\rangle$, but with the trace insertions replaced by ordinary interaction vertices, so it is given (normalized by the free diagram) by (5.8).

5.4 Expansion in topologies

In this section we will develop the large-N expansion for (5.1). The ideas are identical to those of section 4.4. We identify the internal topology of the interacting component (the set of components containing the trace insertions) of each diagram with the corresponding topology in string theory. To justify this, we will show that the leading contribution of all diagrams with topologies of Euler characteristic χ is proportional to N^χ. We will begin by writing expressions for the denominator and the numerator of (5.1), and then use the saddle point method to obtain the desired result for their ratio. The calculation will be very similar to the one in section 4.4, but somewhat more complicated, as there are more classes of diagrams to consider. Here too, we will need to make some assumptions on the existence of certain integration contours in order to use the saddle-point method.

5.4.1 $\langle \det_L(X) \det_L(X^\dagger)\rangle$

Each connected component in a $\langle \det_L(X) \det_L(X^\dagger)\rangle$ diagram can be characterized by k_1, the number of incoming external legs attached to it, k_2, the number of outgoing legs, s, the number of lines which go from X to $X^\dagger$ (recall the discussion of $L_{XX^\dagger}$ in section 5.2), and χ, the Euler characteristic of the internal topology (in the SO$(2N)$ case, the fact that the legs attached to each component must be arranged in an alternating order implied that $k_1 = k_2$, but in the SU(N) theories this need not be the case). Let $D_{k_1,k_2,\chi,s}$ denote the sum of all connected diagrams which form a component of type (k_1, k_2, χ, s), but without the N-dependent factors.

Consider first the sum of all diagrams which have $L_{k_1,k_2,\chi,s}$ connected components of type (k_1, k_2, χ, s). Normalized by the free diagram it is given by:

$$\frac{1}{(N-L)!N^L}(N-S)!S! \prod_{k_1,k_2,\chi,s} \frac{N^{\chi L_{k_1,k_2,\chi,s}} D_{k_1,k_2,\chi,s}^{L_{k_1,k_2,\chi,s}}}{L_{k_1,k_2,\chi,s}!}, \tag{5.10}$$

where $S = \sum_{k_1,k_2,\chi,s} sL_{k_1,k_2,\chi,s}$ is $L_{XX^\dagger}$ in the notation of section 5.2 (recall (5.8)). The product extends over all $k_1, k_2 = 0, \ldots, N$; $\chi = 1, 0, -1, \ldots$; $s = 0, \ldots, \min(k_1, k_2)$. It is easy to see that the factorial terms in the denominators give the right symmetry factors for each diagram in the sum.

To compute $\langle \det_L(X) \det_L(X^\dagger)\rangle$ we sum over all possibilities for $L_{k_1,k_2,\chi,s}$, obtaining

$$\sum_{L_{k_1,k_2,\chi,s}\geq 0} \frac{1}{(N-L)!N^L}(N-S)!S! \prod_{k_1,k_2,\chi,s} \frac{N^{\chi L_{k_1,k_2,\chi,s}} D_{k_1,k_2,\chi,s}^{L_{k_1,k_2,\chi,s}}}{L_{k_1,k_2,\chi,s}!}, \tag{5.11}$$

where the sum is constrained by the two conditions $\sum_{k_1,k_2,\chi,s} k_1 L_{k_1,k_2,\chi,s} = L$ and $\sum_{k_1,k_2,\chi,s} k_2 L_{k_1,k_2,\chi,s} = L$. We will denote the same sum (5.11) with the constraints $\sum_{k_1,k_2,\chi,s} k_1 L_{k_1,k_2,\chi,s} = n$ and $\sum_{k_1,k_2,\chi,s} k_2 L_{k_1,k_2,\chi,s} = m$ by a_{nm}.

5.4.2 $\langle \det_L(X) \det_L(X^\dagger) N \operatorname{tr}(X^{j_1}) N \operatorname{tr}(X^{j_2}) \cdots N \operatorname{tr}(X^{\dagger m_1}) N \operatorname{tr}(X^{\dagger m_2}) \cdots \rangle$

Turning our attention to correlation functions with insertions, let us evaluate the contribution of all diagrams whose interacting component has Euler characteristic χ_0.[12] Every such diagram is characterized by the values of l_1, l_2 (the number of incoming and outgoing legs of the interacting component), s_0 (the contribution of its interacting component to S), and as before, by $L_{k_1,k_2,\chi,s}$, the number of (insertion free) components of type (k_1, k_2, χ, s). These satisfy the two conditions $\sum_{k_1,k_2,\chi,s} k_1 L_{k_1,k_2,\chi,s} + l_1 = L$ and $\sum_{k_1,k_2,\chi,s} k_2 L_{k_1,k_2,\chi,s} + l_2 = L$.

Let $C_{l_1,l_2,\chi,s}$ denote the sum of all diagrams of type (l_1, l_2, χ, s) which include the trace insertions but with no N dependence. The contribution of all diagrams that have specific values of l_1, l_2, s_0 and $L_{k_1,k_2,\chi,s}$ is

$$\frac{1}{(N-L)!N^L} N^{\chi_0} C_{l_1,l_2,\chi_0,s_0} (N-S-s_0)!(S+s_0)! \prod_{k_1,k_2,\chi,s} \frac{N^{\chi L_{k_1,k_2,\chi,s}} D_{k_1,k_2,\chi,s}^{L_{k_1,k_2,\chi,s}}}{L_{k_1,k_2,\chi,s}!} . \tag{5.12}$$

The total contribution comes from summing over all values of l_1, l_2, s_0 and $L_{k_1,k_2,\chi,s}$:

$$\sum_{L_{k_1,k_2,\chi,s}; l_1, l_2, s_0} \frac{N^{\chi_0} C_{l_1,l_2,\chi_0,s_0}}{(N-L)!N^L} (N-S-s_0)!(S+s_0)! \prod_{k_1,k_2,\chi,s} \frac{N^{\chi L_{k_1,k_2,\chi,s}} D_{k_1,k_2,\chi,s}^{L_{k_1,k_2,\chi,s}}}{L_{k_1,k_2,\chi,s}!} , \tag{5.13}$$

where the sum is constrained by the two constraints $\sum_{k_1,k_2,\chi,s} k_1 L_{k_1,k_2,\chi,s} + l_1 = L$ and $\sum_{k_1,k_2,\chi,s} k_2 L_{k_1,k_2,\chi,s} + l_2 = L$. We will denote the same sum (5.13) with the constraints $\sum_{k_1,k_2,\chi,s} k_1 L_{k_1,k_2,\chi,s} + l_1 = n$ and $\sum_{k_1,k_2,\chi,s} k_2 L_{k_1,k_2,\chi,s} + l_2 = m$ by b_{nm}.

5.4.3 Saddle point evaluation

We want to show that the normalized correlation function, the ratio $\frac{(5.13)}{(5.11)} = \frac{b_{LL}}{a_{LL}}$, is proportional to N^{χ_0}. We would like to evaluate it using the saddle point method, as in section 4.4. One complication in this theory is that the appearance of the $S!(N-S)!$ factors makes it more difficult to get a useful form for the partition functions. Using the Beta function, however, we will be able to write the partition function as an integral of a simple exponential function of the kind we obtained in section 4.4, at the price of needing to perform an additional saddle point integration.

Another complication arises from the fact that the number of incoming and outgoing legs of a component need not be equal. Thus, we have two conditions in each sum ((5.11) and (5.13)), so the partition function will have two arguments and a double integration will be necessary. We therefore define the 'partition functions':

$$Z_1(\beta, \gamma) \equiv \sum_{n,m=0}^{\infty} a_{nm} \beta^n \gamma^m , \qquad Z_2(\beta, \gamma) \equiv \sum_{n,m=0}^{\infty} b_{nm} \beta^n \gamma^m . \tag{5.14}$$

From the definition of the Beta function it follows that

$$S!(N-S)! = (N+1)! B(S+1, N-S+1) = (N+1)! \int_0^1 dx x^S (1-x)^{N-S} . \tag{5.15}$$

[12] As in the SO(N) theory, the interacting 'component' may be composed of several disconnected components.

Inserting this into (5.11), (5.13) and (5.14) we get

$$\begin{aligned}
Z_1(\beta,\gamma) &= \sum_{L_{k_1,k_2,\chi,s}\geq 0} \frac{1}{(N-L)!N^L}\left(N-\sum_{k_1,k_2,\chi,s} sL_{k_1,k_2,\chi,s}\right)!\left(\sum_{k_1,k_2,\chi,s} sL_{k_1,k_2,\chi,s}\right)!\cdot \\
&\quad \cdot \prod_{k_1,k_2,\chi,s} \frac{N^{\chi L_{k_1,k_2,\chi,s}}\beta^{k_1 L_{k_1,k_2,\chi,s}}\gamma^{k_2 L_{k_1,k_2,\chi,s}} D_{k_1,k_2,\chi,s}^{L_{k_1,k_2,\chi,s}}}{L_{k_1,k_2,\chi,s}!} \\
&= \int_0^1 dx \sum_{L_{k_1,k_2,\chi,s}\geq 0} \frac{(N+1)!}{(N-L)!N^L} x^N \left(\frac{1-x}{x}\right)^{(\sum_{k_1,k_2,\chi,s} sL_{k_1,k_2,\chi,s})}\cdot \\
&\quad \cdot \prod_{k_1,k_2,\chi,s} \frac{N^{\chi L_{k_1,k_2,\chi,s}}\beta^{k_1 L_{k_1,k_2,\chi,s}}\gamma^{k_2 L_{k_1,k_2,\chi,s}} D_{k_1,k_2,\chi,s}^{L_{k_1,k_2,\chi,s}}}{L_{k_1,k_2,\chi,s}!} \\
&= \frac{(N+1)!}{(N-L)!N^L}\int_0^1 dx\ x^N e^{\sum_{k_1,k_2,\chi,s} N^\chi D_{k_1,k_2,\chi,s}\beta^{k_1}\gamma^{k_2}(\frac{1-x}{x})^s}, \qquad (5.16)
\end{aligned}$$

and similarly

$$Z_2(\beta,\gamma) = \frac{(N+1)!}{(N-L)!N^L} N^{\chi_0}\int_0^1 dx\ x^N e^{\sum_{k_1,k_2,\chi,s} N^\chi D_{k_1,k_2,\chi,s}\beta^{k_1}\gamma^{k_2}(\frac{1-x}{x})^s} g_{\chi_0}(\beta,\gamma,x)\,, \qquad (5.17)$$

where

$$g_{\chi_0}(\beta,\gamma,x) \equiv \sum_{l_1,l_2,s_0} C_{l_1,l_2,\chi_0,s_0}\beta^{l_1}\gamma^{l_2}\left(\frac{1-x}{x}\right)^{s_0}. \qquad (5.18)$$

The coefficients a_{LL} and b_{LL} are given by the double contour integrals:

$$\begin{aligned}
a_{LL} &= -\frac{1}{4\pi^2}\oint d\beta \oint d\gamma\ Z_1(\beta,\gamma)\beta^{-(L+1)}\gamma^{-(L+1)}, \\
b_{LL} &= -\frac{1}{4\pi^2}\oint d\beta \oint d\gamma\ Z_2(\beta,\gamma)\beta^{-(L+1)}\gamma^{-(L+1)}. \qquad (5.19)
\end{aligned}$$

We will begin by analyzing a_{LL}. Recall that we are interested in L's such that $(N-L)$ remains fixed as $N\to\infty$, so we will treat $(N-L)$ as an object of order 1. Keeping only the two leading (first and zeroth) powers of N in the exponent of $Z_1(\beta,\gamma)$ (the leading term cannot vanish since the theory has a propagator) we get

$$a_{LL} \simeq -\frac{1}{4\pi^2}\frac{(N+1)!}{(N-L)!N^L}\oint \frac{d\beta}{\beta^{(L-N+1)}}\oint \frac{d\gamma}{\gamma^{(L-N+1)}}\int_0^1 dx e^{f_0(\beta,\gamma,x)}x^N e^{Nf(\beta,\gamma,x)}, \qquad (5.20)$$

where we define $f(\beta,\gamma,x)\equiv \sum_{k_1,k_2,s}\beta^{k_1}\gamma^{k_2}(\frac{1-x}{x})^s D_{k_1,k_2,1,s} - \log(\beta) - \log(\gamma)$ to include the leading terms and $f_0(\beta,\gamma,x) \equiv \sum_{k_1,k_2,s}\beta^{k_1}\gamma^{k_2}(\frac{1-x}{x})^s D_{k_1,k_2,0,s}$ to include the subleading terms; both functions are independent of N.

We will assume that we may exchange the order of integration, and for each x we perform a saddle point integration on β and γ. Starting with the β integration, we are making the assumption, as in section 4.4, that we can always (at least for generic values of

x and γ) find a contour of fixed imaginary value encircling the origin which goes through a leading saddle point, say $\beta_0(\gamma, x)$. The β integration then gives

$$a_{LL} \simeq -\frac{1}{4\pi^2}\frac{(N+1)!}{(N-L)!N^L}\int_0^1 dx x^N \oint \frac{d\gamma}{\gamma^{(L-N+1)}}\frac{e^{i\alpha(\gamma,x)}\sqrt{2\pi}e^{N|f(\beta_0(\gamma,x),\gamma,x)|}e^{f_0(\beta_0(\gamma,x),\gamma,x)}}{\sqrt{2N|\partial_\beta^2 f(\beta_0(\gamma,x),\gamma,x)|}\beta_0(\gamma,x)^{L-N+1}}, \tag{5.21}$$

where $e^{i\alpha}$ is a phase determined by the direction of steepest descent. Similarly, assuming that we can find a steepest descent contour around the origin for the γ integration we find

$$\begin{aligned} a_{LL} \simeq & -\frac{1}{4\pi^2}\frac{(N+1)!}{(N-L)!N^L}\int_0^1 dx x^N e^{i\tilde{\alpha}(x)}\frac{1}{\beta_0(\gamma_0(x),x)^{L-N+1}\gamma_0(x)^{L-N+1}} \cdot \\ & \cdot \frac{2\pi e^{N|f(\beta_0(\gamma_0(x),x),\gamma_0(x),x)|}e^{f_0(\beta_0(\gamma_0(x),x),\gamma_0(x),x)}}{2N\sqrt{|\partial_\beta^2 f(\beta_0(\gamma_0(x),x),\gamma_0(x),x)\partial_\gamma^2 f(\beta_0(\gamma_0(x),x),\gamma_0(x),x)|}}. \end{aligned} \tag{5.22}$$

The phase $e^{i\tilde{\alpha}}$ in the last expression actually makes it real. Indeed, working in the euclidean theory, all $D_{k_1,k_2,\chi,s}$ can be taken to be real, and for each fixed x, the β and γ integrations just give a coefficient in the power series of a real function. The important thing about the x integral in (5.22) is that it can be written in the form $\int_0^1 dx u(x)e^{Nv(x)}$ where u and v are real. This means, since $N \gg 1$, that practically the only contribution will come from a very narrow part of the unit interval surrounding the point for which $v(x)$ is maximal (or possibly diverging), say x_0. The exact calculation depends on whether x_0 is inside the interval or on the endpoints, and on whether $u(x_0)$ vanishes or not (and, if it vanishes, of what order is the zero), but this is not important for our purposes. What is important is that when we make the same analysis for (5.13) $= b_{LL}$, the only difference (apart from a factor of N^{χ_0}) is the appearance of the N-independent function $g_{\chi_0}(\beta,\gamma,x)$ in the integrand. When we perform the β and γ integrations the result will just have a factor of $g_{\chi_0}(x) \equiv g_{\chi_0}(\beta_0(\gamma_0(x),x),\gamma_0(x),x)$ in front of it. So, we get

$$\begin{aligned} (5.11) = a_{LL} &\simeq \frac{(N+1)!}{(N-L)!N^L}\int_0^1 dx u(x)e^{Nv(x)}, \\ (5.13) = b_{LL} &\simeq N^{\chi_0}\frac{(N+1)!}{(N-L)!N^L}\int_0^1 dx g_{\chi_0}(x)u(x)e^{Nv(x)} \\ &\simeq N^{\chi_0}\frac{(N+1)!}{(N-L)!N^L}g_{\chi_0}(x_0)\int_0^1 dx u(x)e^{Nv(x)}, \end{aligned} \tag{5.23}$$

where the last equality comes from the fact that the only contribution to the integral comes from the very narrow neighborhood around x_0.[13]

For the ratio we are interested in we simply get

$$\frac{(5.13)}{(5.11)} \simeq g_{\chi_0}(x_0)N^{\chi_0}, \tag{5.24}$$

[13]Here we are assuming that $g_{\chi_0}(x_0) \neq 0$. We see no reason why this should not hold for a generic correlation function and for generic 't Hooft coupling. When $g_{\chi_0}(x)$ does have a zero of order f at x_0, the contribution of this topology to the ratio we are interested in will be of order $N^{\chi_0 - f}$ instead of N^{χ_0}, and the leading term of order N^{χ_0} vanishes.

which has precisely the expected dependence on N. Thus, up to some assumptions about the existence of appropriate contours, we have shown that in any $\mathrm{SU}(N)$ gauge theory (5.1) has a topological expansion involving both closed and open string worldsheets.

We have not been able to rigorously justify the contour integrals performed above, but appropriate contours exist in all examples which we explicitly checked. There is a special case where the contour integrations simplify, which is when the theory has a U(1) symmetry taking $X \to e^{i\alpha} X$ (this is true in the $\mathcal{N} = 4$ SYM theory). In such theories, each component with no insertions must have $k_1 = k_2$, as in the $\mathrm{SO}(2N)$ case. All the formulas can then be rewritten as a single integral, as in section 4, and we find the same result (5.24). In this special case, after performing the integration over β, the contour integral over γ in (5.21) is actually trivial (the γ dependence is exactly $1/\gamma$) so we do not need to do a saddle-point integral for γ, but in generic theories we expect that both saddle point integrations will be required.

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References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[5] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[6] E. Witten, *Baryons and branes in anti de Sitter space*, *J. High Energy Phys.* **07** (1998) 006 [hep-th/9805112].

[7] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *J. High Energy Phys.* **06** (2000) 008 [hep-th/0003075].

[8] R.C. Myers, *Dielectric-branes*, *J. High Energy Phys.* **12** (1999) 022 [hep-th/9910053].

[9] V. Balasubramanian, M. Berkooz, A. Naqvi and M.J. Strassler, *Giant gravitons in conformal field theory*, *J. High Energy Phys.* **04** (2002) 034 [hep-th/0107119].

[10] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and PP waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[11] V. Balasubramanian, M.-x. Huang, T.S. Levi and A. Naqvi, *Open strings from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **08** (2002) 037 [hep-th/0204196].

[12] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, *Linear dilatons, NS5-branes and holography*, *J. High Energy Phys.* **10** (1998) 004 [hep-th/9808149].

[13] E. Witten, *Baryons in the $1/N$ expansion*, *Nucl. Phys.* **B 160** (1979) 118.

[14] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual $N = 4$ sym theory*, *Adv. Theor. Math. Phys.* **5** (2002) 809 [hep-th/0111222];
S. Corley and S. Ramgoolam, *Finite factorization equations and sum rules for BPS correlators in $N = 4$ sym theory*, *Nucl. Phys.* **B 641** (2002) 131 [hep-th/0205221].

[15] S.S. Gubser and I.R. Klebanov, *Baryons and domain walls in an $N = 1$ superconformal gauge theory*, *Phys. Rev.* **D 58** (1998) 125025 [hep-th/9808075].

[16] D.R. Morrison and M.R. Plesser, *Non-spherical horizons 1*, *Adv. Theor. Math. Phys.* **3** (1999) 1 [hep-th/9810201].

[17] S. Gukov, M. Rangamani and E. Witten, *Dibaryons, strings and branes in AdS orbifold models*, *J. High Energy Phys.* **12** (1998) 025 [hep-th/9811048].

[18] D. Berenstein, C.P. Herzog and I.R. Klebanov, *Baryon spectra and AdS/CFT correspondence*, *J. High Energy Phys.* **06** (2002) 047 [hep-th/0202150].

[19] C.E. Beasley, *BPS branes from baryons*, *J. High Energy Phys.* **11** (2002) 015 [hep-th/0207125].

[20] M.T. Grisaru, R.C. Myers and O. Tafjord, *SUSY and Goliath*, *J. High Energy Phys.* **08** (2000) 040 [hep-th/0008015].

[21] A. Hashimoto, S. Hirano and N. Itzhaki, *Large branes in AdS and their field theory dual*, *J. High Energy Phys.* **08** (2000) 051 [hep-th/0008016].

[22] G.M. Cicuta, *Topological expansion for* $\mathrm{SO}(N)$ *and* $\mathrm{Sp}(2N)$ *gauge theories*, *Nuovo Cim.* **35** (1982) 87.

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